# Approximation of functions by polynomials

Why?

Sometimes the analytic expression of a function is complicated, and it might be difficult to integrate it or to differentiate it.

Sometimes the analytic expression is not available, the function is known only at some points at we need to reconstruct the underlyning function.

A way to overcome problems of this kind is to approximate functions with polynomials (easy to deal with).

**Theorem 1**

*Let $g : [c, d] \to \mathbb{R}$ be a smooth enough function. Given $k + 1$ distinct points $x_1, x_2, \cdots, x_{k+1}$ in $[c, d]$, there exists a unique polynomial $\Pi_k(x)$ of degree $\leq k$ such that*

$$\Pi_k(x_i) = g(x_i) \qquad i = 1, 2, \cdots, k + 1. \qquad (*)$$

*$\Pi_k$ is called "Lagrange interpolant of $g$ with respect to the points $x_1, x_2, \cdots, x_{k+1}$".*
*The points $x_1, x_2, \cdots, x_{k+1}$ are called "interpolation nodes".*

## Proof of Existence.

We shall write explicitely the expression of $\Pi_k$.
For this, let $L_i(x)$, $i = 1, 2, \cdots, k+1$ be $k+1$ polynomials, of degree exactly $k$, defined by:

$$L_i(x) = \prod_{\substack{j=1 \\ j \neq i}}^{k+1} \frac{(x - x_j)}{(x_i - x_j)} \quad i = 1, 2, \cdots, k+1$$

For example, for $k = 3$ and nodes $x_1, x_2, x_3, x_4$:

$$L_1(x) = \frac{(x - x_2)(x - x_3)(x - x_4)}{(x_1 - x_2)(x_1 - x_3)(x_1 - x_4)}$$

$$L_2(x) = \frac{(x - x_1)(x - x_3)(x - x_4)}{(x_2 - x_1)(x_2 - x_3)(x_2 - x_4)}$$

$$L_3(x) = \frac{(x - x_1)(x - x_2)(x - x_4)}{(x_3 - x_1)(x_3 - x_2)(x_3 - x_4)}$$

$$L_4(x) = \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_4 - x_1)(x_4 - x_2)(x_4 - x_3)}$$

□

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These are characteristic Lagrange polynomials, that fulfil

$$L_i \in \mathbb{P}_k : \quad L_i(x_j) = \delta_{ij} = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases} \quad i = 1, 2, \cdots .k+1$$

Claim:

$$\Pi_k(x) := \sum_{i=1}^{k+1} g(x_i) L_i(x)$$

Indeed, it is easy to check that $\Pi_k$ verifies $(*)$, and this proves **existence**.

□

## Proof of uniqueness.

The problem of finding the polynomial

$$\Pi_k(x) := \alpha_1 x^k + \ldots + \alpha_k x + \alpha_{k+1}$$

that interpolate $g$ at the $k+1$ distinct nodes $x_1, x_2, \cdots, x_{k+1}$ is a linear problem, that takes the form:

$$\begin{bmatrix} x_1^k & x_1^{k-1} & \ldots & 1 \\ x_2^k & x_2^{k-1} & \ldots & 1 \\ \vdots & \vdots & \ldots & 1 \\ x_{k+1}^k & x_{k+1}^{k-1} & \ldots & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{k+1} \end{bmatrix} = \begin{bmatrix} g(x_1) \\ g(x_2) \\ \vdots \\ g(x_{k+1}) \end{bmatrix}$$

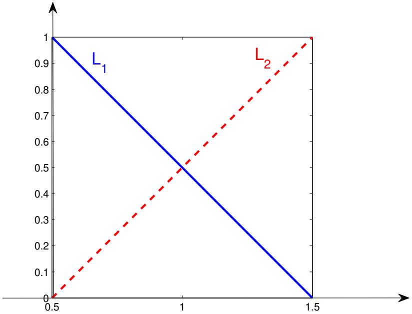
The linear system has dimension $k+1 \times k+1$. Since the problem has always a solution (existence is proved in the previous slide) then the solution is unique . □

# Example of characteristic polynomials

Ex. 1: 2 points $x_1 \neq x_2$ $(k+1=2)$

$$L_1(x) = \frac{(x - x_2)}{(x_1 - x_2)}, \quad L_2(x) = \frac{(x - x_1)}{(x_2 - x_1)}$$

degree $k = 1$.



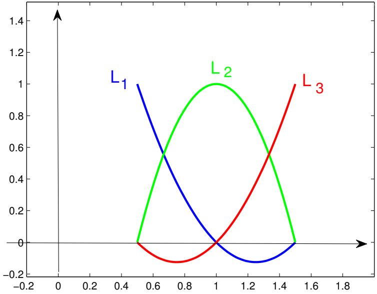
# Example of characteristic polynomials

Ex. 2: 3 points $x_1 \neq x_2 \neq x_3$ $(k+1=3)$

$$L_1(x) = \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)}, \quad L_2(x) = \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)}$$

$$L_3(x) = \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)}$$

degree $k = 2$.



# Cases of interest

Case 1: $g$ approximated by a constant (a polynomial of degree 0)
1 point: $x_1 = \frac{c+d}{2}$ (midpoint of the interval)

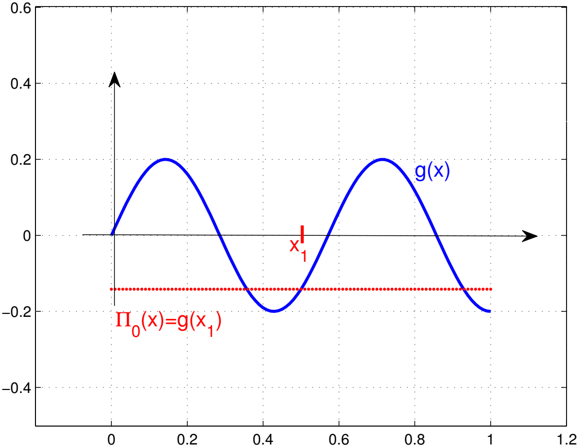
$$g(x) \simeq \Pi_0(x) = g(x_1)$$

Case 2: $g$ approximated by a polynomial of degree 1(straight line)
2 point: $x_1 = c$, $x_2 = d$

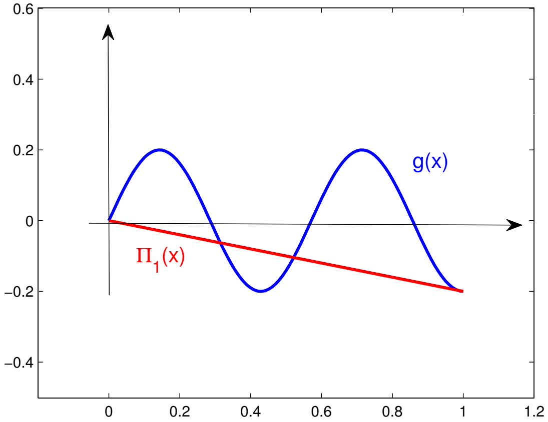
$$\begin{aligned} g(x) \simeq \Pi_1(x) &= g(x_1)L_1(x) + g(x_2)L_2(x) \\ &= g(x_1)\frac{(x-x_2)}{(x_1-x_2)} + g(x_2)\frac{(x-x_1)}{(x_2-x_1)} \end{aligned}$$

Remark: in both cases, if we change the nodes we obtain a different interpolant

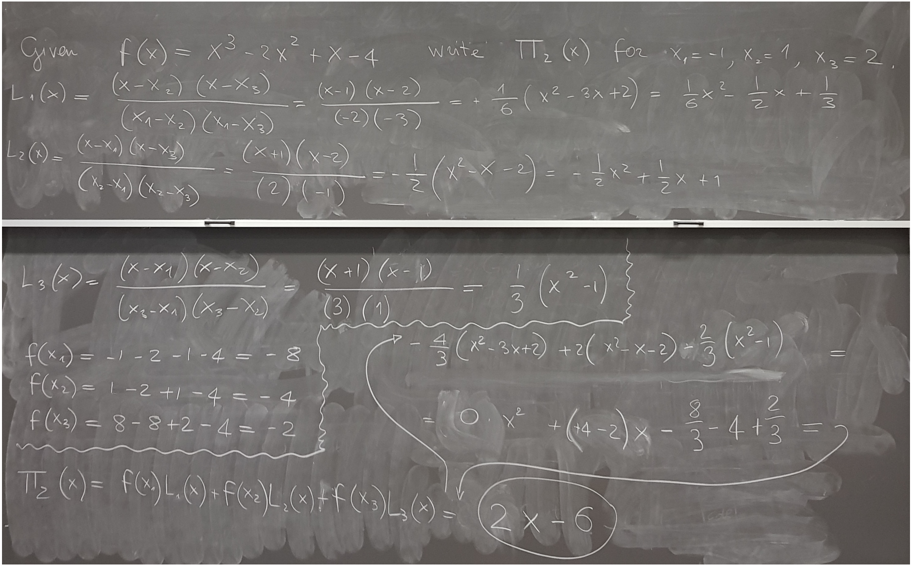
# Example 1



# Example 2



# Exercise


Given f(x) = x^3 - 2x^2 + x - 4 write Π_2 (x) for x_1 = -1, x_2 = 1, x_3 = 2.
L_1(x) = (x - x_2)(x - x_3) / (x_1 - x_2)(x_1 - x_3) = (x-1)(x-2) / (-2)(-3) = + 1/6 (x^2 - 3x + 2) = 1/6 x^2 - 1/2 x + 1/3
L_2(x) = (x - x_1)(x - x_3) / (x_2 - x_1)(x_2 - x_3) = (x+1)(x-2) / (2)(-1) = - 1/2 (x^2 - x - 2) = - 1/2 x^2 + 1/2 x + 1
L_3(x) = (x - x_1)(x - x_2) / (x_3 - x_1)(x_3 - x_2) = (x+1)(x-1) / (3)(1) = 1/3 (x^2 - 1)
f(x_1) = -1 - 2 - 1 - 4 = - 8
f(x_2) = 1 - 2 + 1 - 4 = - 4
f(x_3) = 8 - 8 + 2 - 4 = - 2
- 4/3 (x^2 - 3x + 2) + 2 (x^2 - x - 2) - 2/3 (x^2 - 1) =
= 0 · x^2 + (+4 - 2) x - 8/3 - 4 + 2/3 =
Π_2 (x) = f(x_1) L_1(x) + f(x_2) L_2(x) + f(x_3) L_3(x) = 2x - 6

# General interpolation error

In both cases, the approximation induces an error that we want to estimate. How big is it? Is the approximation satisfactory? The following Theorem gives a precise expression of the error.

## Theorem 2

*Let $g \in C^{k+1}([c,d])$, and let $x_1, x_2, \cdots, x_{k+1}$ be $k+1$ distinct points in $[c,d]$. For any generic point $x \in [c,d]$ there exists a $\xi \in [c,d]$ (depending on $x$) such that the interpolation error is given by*

$$e_k(x) := g(x) - \Pi_k(x) = \frac{g^{(k+1)}(\xi)}{(k+1)!} \prod_{j=1}^{k+1} (x - x_j) \tag{1}$$

## Proof * NOT FOR THE EXAM *.

Let $x$ be fixed and $t \in [c, d]$. Define the function $\omega_{k+1}(t) = \prod_{j=1}^{k+1}(t - x_j)$ let us introduce

$$G(t) := e_k(t) - \omega_{k+1}(t)e_k(x)/\omega_{k+1}(x).$$

Since $g \in C^{k+1}([c, d])$ and $\omega_{k+1}$ is a polynomial, $G \in C^{k+1}([c, d])$ and vanishes at the $k + 2$ distinct points $x_1, x_2, \cdots, x_{k+1}$ and $x$. Indeed:

$$G(x_i) = e_k(x_i) - \omega_{k+1}(x_i)e_k(x)/\omega_{k+1}(x) = 0 \quad i = 1, 2, \cdots, k+1$$
$$G(x) = e_k(x) - \omega_{k+1}(x)e_k(x)/\omega_{k+1}(x) = 0$$

Thanks to the meanvalue theorem[a], $G'$ will have $k + 1$ distinct zeros, and going recursively, $G^{(j)}$ will have $k + 2 - j$ distinct zeros. Hence, $G^{(k+1)}$ will have one zero, say $\xi$. Since $\omega_{k+1}^{(k+1)}(t) = (k + 1)!$ and $e_k^{(k+1)}(t) = g^{(k+1)}(t)$ we obtain the result

$$G^{(k+1)}(\xi) = g^{(k+1)}(\xi) - (k + 1)!e_k(x)/\omega_{k+1}(x) = 0$$

and the proof is concluded. □

[a] https://en.wikipedia.org/wiki/Mean_value_theorem

From (1) we can deduce for instance the following bound:

$$\max_{[c,d]} |e_k(x)| \leq \frac{(d-c)^{k+1}}{(k+1)!} \max_{x\in[c,d]} |g^{(k+1)}(x)| \qquad (*)$$

If we know the position of the nodes we can have sharper estimates for the term $\prod_{j=1}^{k+1}(x - x_j)$ that appears in (1). For instance, for the Case 1 we would obtain

Case 1 (constant approximation): Relation (1) in this case gives

$$e_0(x) = g'(\xi)(x - x_1) \quad x_1 = \frac{c+d}{2}$$

No matter where $x$ is situated within $[c, d]$, its distance from the midpoint will be smaller than or equal to the length of half the interval (that is, $(d-c)/2$). Hence

$$\implies \quad \max_{x\in[c,d]} |e_0(x)| \leq \frac{d-c}{2} \max_{x\in[c,d]} |g'(x)| \qquad (2)$$

($(d-c)/2$ instead of $d-c$)

Case 2 (linear approximation): For $x_1 = c$ and $x_2 = d$ we can observe that the function $x \to |(x-c)(x-d)|$, in the interval $[c, d]$ has its maximum for $x = (c+d)/2$ and such a maximum is $((d-c)/2)^2$. Hence

$$\max_{x \in [c,d]} |e_1(x)| \leq \frac{(d-c)^2}{2! \cdot 4} \max_{x \in [c,d]} |g''(x)| \tag{3}$$

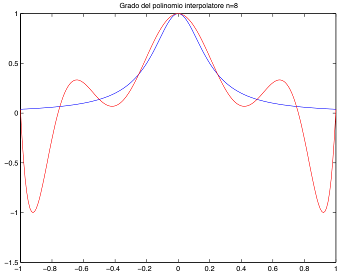
(and we have $(d-c)^2/8$ instead of $(d-c)^2/2$).

# Runge function

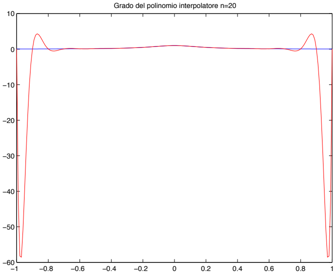
Apparently, increasing the degree $k$ of the Lagrange polynomial should improve the error, which should become smaller and smaller. This is not always the case if the nodes are equally spaced. The classical example is given by the so-called Runge's function:

$$g(x) = \frac{1}{1 + x^2}$$

This is what happens




degree 8




degree 20

# A simple remedy: Composite Lagrange interpolation

Use of Lagrange interpolation (piecewise) to have a good approximation of a function.

Given $f : [a, b] \to \mathbb{R}$ (smooth enough), subdivide $[a, b]$ in $N$ subintervals, for simplicity of notation all equal. We have then a uniform subdivision of $[a, b]$ into intervals of length $h = (b - a)/N$. In each subinterval we approximate $f$ with a Lagrange interpolant polynomial.

To fix ideas, let us consider again Cases 1 and 2.

Case 1: $N \to h = (b-a)/N, \quad x_1 = a, x_2 = x_1 + h, \cdots, x_{N+1} = b$

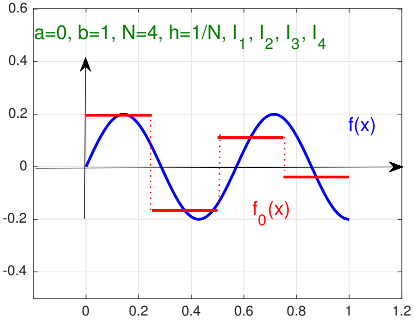
$I_1 = [x_1, x_2], \cdots, I_j = [x_j, x_{j+1}], \cdots, I_N = [x_N, x_{N+1}]$

$x_j^M$ =midpoint of the interval $I_j$ : $x_j^M = (x_j + x_{j+1})/2$

$f(x)_{|[a,b]} \simeq f_0(x)$ piecewise constant function given by

$$f_0(x)_{|I_j} = f(x_j^M) \quad j = 1, 2, \cdots, N$$

# Piecewise constant interpolation



Let us study the error $E_0(x) := f(x) - f_0(x) \quad x \in [a, b]$
The maximum error (in absolute value) will be achieved on one subinterval, say $\bar{I}_k$. Then,

$$\max_{x\in[a,b]} |E_0(x)| = \max_{x\in[a,b]} |f(x) - f_0(x)| = \max_{x\in\bar{I}_k} |f(x) - f_0(x_k^M)|$$

Using the mean value theorem[1], here exists $c$ between $x$ and $x_k^M$ such that:

$$\frac{f(x) - f_0(x_k^M)}{x - x_k^M} = f'(c)$$

Then $f(x) - f_0(x_k^M) = (x - x_k^M)f'(c)$ and[2]

$$\max_{x\in[a,b]} |E_0(x)| = \max_{x\in\bar{I}_k} |f(x) - f_0(x_k^M)| \leq \frac{h}{2} \max_{c\in\bar{I}_k} |f'(c)| \leq \frac{h}{2} \max_{c\in[a,b]} |f'(c)|$$

If $f'$ exists and is bounded, the interpolation error goes to zero as $C\,h$

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[1] https://en.wikipedia.org/wiki/Mean_value_theorem

[2] Or, use (2), with $[c, d] = [x_i, x_{i+1}] \Longrightarrow d - c = h$ to get the blue estimate

Case 2: $N \to h = (b-a)/N, \quad x_1 = a, x_2 = x_1 + h, \cdots, x_{N+1} = b$

$I_1 = [x_1, x_2], \cdots, I_j = [x_j, x_{j+1}], \cdots, I_N = [x_N, x_{N+1}]$

$f(x)_{|[a,b]} \simeq f_1(x)$ piecewise linear function which, on each interval $I_j$, is the Lagrange interpolant of degree $\leq 1$ with respect to the endpoints of $I_j$, i.e.,

$$f_1(x)_{|I_j} = f(x_j)\frac{(x - x_{j+1})}{(x_j - x_{j+1})} + f(x_{j+1})\frac{(x - x_j)}{(x_{j+1} - x_j)} \quad j = 1, 2, \cdots, N$$

Using the bound (3) for the error $E_1(x) = f(x) - f_1(x)$ and proceeding as before we then obtain

$$\boxed{\max_{[a,b]} |E_1(x)| \leq \frac{h^2}{8} \max_{[a,b]} |f''(x)| = C\, h^2 \qquad C = \frac{\max_{[a,b]} |f''(x)|}{8}}$$

Hence: if $f''$ exists and is bounded, the interpolation error goes to zero quadratically with $h$ (if you halve $h$ the error is divided by four), and we can choose $h$ as small as we want!

# Piecewise linear interpolation